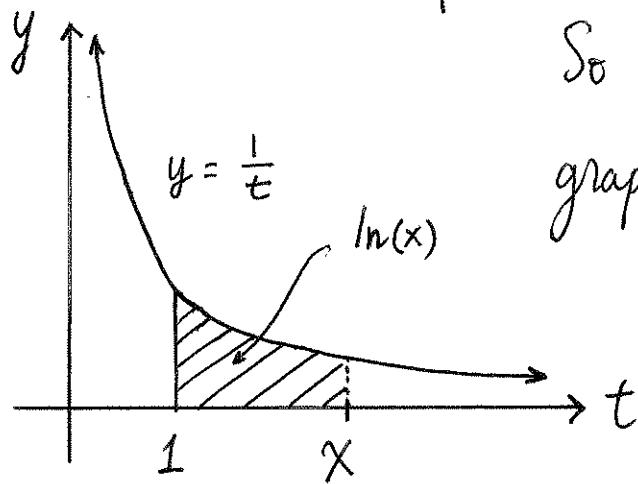


## Section 6.2: The natural logarithmic function

**Definition:** The natural log function is defined by

$$\ln(x) = \int_1^x \frac{1}{t} dt, \text{ for any } x > 0.$$



So  $\ln(x)$  is the Area between the graph of  $\frac{1}{t}$  and  $y=0$ , between  $t=1$  and  $t=x$ .

- if  $x > 1$ ,  $\ln(x) > 0$
- if  $x < 1$ ,  $\ln(x) = \int_1^x \frac{1}{t} dt = - \int_x^1 \frac{1}{t} dt < 0$
- if  $x = 1$ ,  $\ln(1) = 0$ .

**Differentiation:** By the fundamental theorem of Calculus,

$$\frac{d}{dx} \ln(x) = \frac{d}{dx} \left( \int_1^x \frac{1}{t} dt \right) = \frac{1}{x}, \text{ for any } x > 0.$$

Laws of The natural log ( $\ln$ ).

Let  $x, y > 0$ , and  $r$  be a rational number. Then

$$\textcircled{1} \quad \ln(x \cdot y) = \ln(x) + \ln(y)$$

$$\textcircled{2} \quad \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$\textcircled{3} \quad \ln(x^r) = r \ln(x)$$

Proof: (1) Let  $f(x) = \ln(ax)$ , where  $a$  is a constant.

By Chain rule,  $f'(x) = \frac{1}{ax} \cdot a = \frac{1}{x}$ . So,

$$\frac{d}{dx} (\ln(x)) = \frac{d}{dx} (\ln(ax)). \quad \text{This means that}$$

$$\ln(ax) = \ln(x) + C, \quad \text{for some constant } C. \quad \text{Let } x=1;$$

$$\text{Then } \ln(a) = \ln(1) + C = C. \quad \text{Therefore,}$$

$$\ln(ax) = \ln(x) + \ln(a).$$

$$\textcircled{2} \quad \ln\left(\frac{x}{y}\right) = \ln(x \cdot y^{-1}) = \ln(x) + \ln\left(\frac{1}{y}\right).$$

By law ①,

$$0 = \ln(1) = \ln\left(\frac{1}{y} \cdot y\right) = \ln\left(\frac{1}{y}\right) + \ln(y), \text{ and so,}$$

$$\ln\left(\frac{1}{y}\right) = -\ln(y);$$

$$\text{Therefore, } \ln\left(\frac{x}{y}\right) = \ln(x) + \ln\left(\frac{1}{y}\right) = \ln(x) - \ln(y)$$

Example: Simplify as much as possible:

$$\ln \frac{(2x^3-1)^3 \tan x}{\sqrt{x+1}}$$

$$\begin{aligned}
 \ln \frac{(2x^3-1)^3 \tan x}{\sqrt{x+1}} &= \ln[(2x^3-1)^3 \tan x] - \ln[\sqrt{x+1}] \\
 &= \ln[(2x^3-1)^3] + \ln(\tan x) - \ln[(x+1)^{\frac{1}{2}}] \\
 &= 3 \ln(2x^3-1) + \ln(\tan x) - \frac{1}{2} \ln(x+1).
 \end{aligned}$$

Limits

$$\textcircled{1} \lim_{x \rightarrow \infty} \ln(x) = \infty, \quad \textcircled{2} \lim_{x \rightarrow 0^+} \ln(x) = -\infty$$

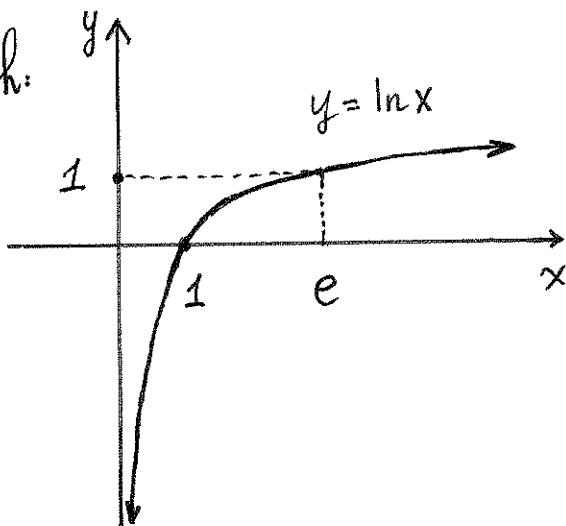
proof: we will only prove \textcircled{2}, and assume \textcircled{1} is true.

Let  $t = \frac{1}{x}$ . if  $x \rightarrow 0^+$ , Then  $t \rightarrow +\infty$ .

$$\lim_{x \rightarrow 0^+} \ln(x) = \lim_{t \rightarrow +\infty} \ln\left(\frac{1}{t}\right) = \lim_{t \rightarrow +\infty} (-\ln(t)) = -\infty.$$

Concavity: If  $f(x) = \ln(x)$ , Then  $f'(x) = \frac{1}{x}$ , and  $f''(x) = -\frac{1}{x^2} < 0$  for all  $x > 0$ . This means that the graph of  $\ln(x)$  is concave down for all  $x > 0$

Graph:



- $e$  is the number such that  $\ln(e) = 1$
- $e$  is irrational;
- $e \approx 2.71828\dots$

Chain Rule: if  $u$  is a function of  $x$ , Then

$$\frac{d}{dx} \ln(u) = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}.$$

Examples:  $\frac{d}{dx} \ln(x^4 - x^2) = \frac{1}{x^4 - x^2} \cdot (4x^3 - 2x)$

$$\frac{d}{dx} \ln(\tan x) = \frac{1}{\tan(x)} \cdot \sec^2(x)$$

Example: Find  $f'(x)$  if  $f(x) = \ln|x|$ ,  $x \neq 0$

We have  $f(x) = \ln|x| = \begin{cases} \ln x, & \text{if } x > 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}$

So,  $f'(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x}, & \text{if } x < 0 \end{cases}$

Therefore,  $\frac{d}{dx} \ln|x| = \frac{1}{x}$ ; This means that

$$\int \frac{1}{x} dx = \ln|x| + C.$$

Example: Find  $\int \frac{3x^2+1}{x^3+x} dx$ . Let  $u = x^3 + x$ ,  
 $du = (3x^2+1) dx$

$$\int \frac{du}{u} = \ln|u| + C$$

$$= \ln|x^3+x| + C \quad \text{U-substitution}$$

Example:  $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx.$  Let  $u = \sin x$   
 $du = \cos x \, dx$

$$= \int \frac{1}{u} \, du = \ln |u| + C$$

$$= \ln |\sin x| + C.$$

Similarly,  $\int \tan x \, dx = -\ln |\cos x| + C$   
 $= \ln |\sec x| + C$

Logarithmic Differentiation.

Differentiate  $y = \frac{x^{-\frac{1}{2}} \sin(x^2+1)}{(2x-1)^2 \cdot \sqrt{2x}}$

$$\ln(y) = -\frac{1}{2} \ln(x) + \ln(\sin(x^2+1)) - 2 \ln(2x-1) - \frac{1}{2} \ln(2x)$$

Take derivative on both sides, with respect to  $x.$

$$\frac{1}{y} \cdot y' = -\frac{1}{2x} + \frac{\cos(x^2+1) \cdot 2x}{\sin(x^2+1)} - 2 \frac{2}{2x-1} - \frac{1}{2} \cdot \frac{1}{2x} \cdot 2$$

Multiply both sides by  $y = \frac{x^{-\frac{1}{2}} \sin(x^2+1)}{(2x-1)^2 \sqrt{2x}}$  to get  $y'.$